MATHEMATICS

1. $|z-4| = \text{Re}(z) \Rightarrow y^2 = 8(x-2)$

Tangent is $y = m(x-2) + \frac{2}{m} \Rightarrow 0 = -2m + \frac{2}{m} \Rightarrow m = \pm 1 \Rightarrow z_1 = 4 + 4i$ & $z_2 = 4 - 4i$

2. Let $z = a + ib$, $z^{2015} = \bar{z} \Rightarrow |z|^{2015} = |z|$

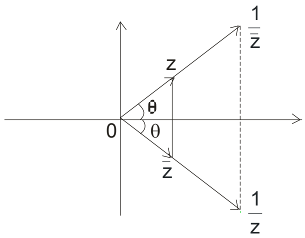
$\Rightarrow |z|\,(|z|^{2014} - 1) = 0 \Rightarrow |z| = 0$ or $|z| = 1$

when $|z| = 0$, $z = 0$

when $|z| = 1$, $z^{2015} = \bar{z} \Rightarrow z^{2016} = z\bar{z} = 1 \Rightarrow$ 2016 roots

$\therefore$ equation $z^{2015} =$ has total 2017 roots.

3. Required area $= \left|\frac{1}{2}.\left|\frac{1}{\bar{z}}\right|.\left|\frac{1}{z}\right|\sin 2\theta - \frac{1}{2}|z||\bar{z}|\sin 2\theta\right|$



$= \frac{1}{2}|\sin 2\theta|.\left|\frac{1}{|z|^2} - |z|^2\right| = \frac{1}{2}.\left|\frac{z^2 - \bar{z}^2}{2i|z|^2}\right|.\left|\frac{1}{|z|^2} - |z|^2\right|$ $\left\{\begin{array}{c} \because\ z - \bar{z} = 2ir\sin\theta \\ z + \bar{z} = 2r\cos\theta \\ \Rightarrow\ z^2 - \bar{z}^2 = 2ir^2\sin 2\theta \end{array}\right\}$

$= \frac{1}{4}|z^2 - \bar{z}^2|\left|\frac{1}{|z|^4} - 1\right|$

4. $(1+\omega)(1+\omega^2) \ldots\ldots (1+\omega^{1988}) = \{(1+\omega)(1+\omega^2)(1+\omega^3)\}^{662}.(1+\omega^{1987})(1+\omega^{1988}) = 2^{662} = 4^{331}$

5. $|z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1| \geq |z_1 + 1| + |(z_2 + 1) - (z_1z_2 + 1)|$

$\geq |z_1 + 1| + |z_2(1 - z_1)|$

$\geq |1 + z_1| + |1 - z_1| \geq 2$

6. Put $x^2y^2 = t \Rightarrow 2xy^2 + 2x^2y.\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \tan t = \frac{dt}{dx} \Rightarrow \int dx = \int \cot t\, dt \Rightarrow x = \ell n|\sin t| + c$

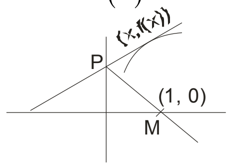
7. $\frac{dy}{y} + \frac{\sin x}{1 + \cos x}dx = 0$

$\Rightarrow \int\frac{dy}{y} + \int\tan\frac{x}{2}dx = c \Rightarrow \ell n|y| + \frac{\ell n\left|\sec\frac{x}{2}\right|}{\frac{1}{2}} = \ell nc \Rightarrow |y|.\sec^2\frac{x}{2} = c$

8. $\frac{y}{x}\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{f(y^2/x^2)}{f'(y^2/x^2)} \Rightarrow \frac{y}{x} = v \Rightarrow \frac{2vf'(v^2)dv}{f(v^2)} = 2\frac{dx}{x} \Rightarrow f(v^2) = cx^2$

9. Equation of tangent $Y - f(x) = f'(x)\,(X - x)$ $\Rightarrow$ $P(0,\ f(x) - xf'(x))$

$m_{PM} = \dfrac{-1}{f'(x)} = \dfrac{f(x) - xf'(x)}{-1}$ $\Rightarrow$ $yy' - x(y')^2 = 1$



10. $f'(x) - \dfrac{2x(x+1)}{(x+1)} f(x) = \dfrac{e^{x^2}}{(x+1)^2}$ I.F. $= e^{-x^2}$ $\Rightarrow$ $f(x)\ e^{-x^2} = \displaystyle\int \dfrac{dx}{(x+1)^2}$

11. Let $\omega = re^{i\theta}$ and $z = x + iy$

$\therefore\ x + iy = re^{i\theta} + \dfrac{e^{-i\theta}}{r}$ $\Rightarrow x = \left(r + \dfrac{1}{r}\right)\cos\theta$ & $y = \left(r - \dfrac{1}{r}\right)\sin\theta$ $\Rightarrow \dfrac{x^2}{\left(r+\frac{1}{r}\right)^2} + \dfrac{y^2}{\left(r-\frac{1}{r}\right)^2} = 1$

Distance between focii $= 2ae = 2\sqrt{\left(r+\dfrac{1}{r}\right)^2 - \left(r-\dfrac{1}{r}\right)^2} = 4$

12. The number of common vertices is given by the number of common roots of $z^{1982} - 1 = 0$ and $z^{2973} - 1 = 0$, which is equal to HCF (1982, 2973) = 991.

13. $z - 1 = e^{i\theta}$ $\Rightarrow$ $z = 2\cos(\theta/2)\, e^{i(\theta/2)}$ $\Rightarrow$ $\tan\left(\arg\dfrac{(z-1)}{2}\right) - \left(\dfrac{2i}{z}\right) = \tan\left(\dfrac{\theta}{2}\right) - \dfrac{i}{\cos\frac{\theta}{2}}\, e^{-i\theta/2} = -i$

14. $\theta_1 - \pi/4 = \theta_2 + 2m\pi$ and $\theta_1 + \theta_2 = 2n\pi + \pi/2$

15. $|x - y| = 4|\cos\theta - \sin\theta| = 4\sqrt{1 - \sin 2\theta} = \left[0,\ 4\sqrt{2}\right]$ (putting $z = 4e^{i\theta}$)

16. $z^3 = t$ $\Rightarrow$ $t = \omega$ or ω^2

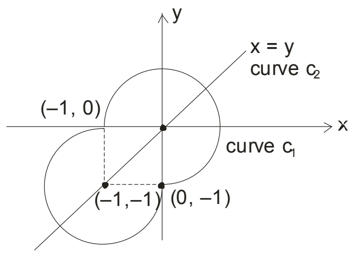
$z = e^{(i)\left[\left(2m\pi + \frac{2\pi}{3}\right)/3\right]}$ or $e^{i\left[\left(2m\pi + \frac{4\pi}{3}\right)/3\right]}$ $\Rightarrow$ $\theta = \dfrac{2m\pi}{3} + \dfrac{2\pi}{9}$ or $\dfrac{2m\pi}{3} + \dfrac{4\pi}{9}$

17. $\dfrac{\bar{z}_1}{z_2} = \dfrac{r_1 e^{-i\theta_1}}{r_2 e^{i\theta_2}} = 2i$ $\therefore$ $z = \dfrac{2i + \omega + \omega^2}{3} = \dfrac{2i - 1}{3}$

18. Equation of line passing through z_1 & z_2 is

$z = z_1 + t(z_2 - z_1)\ ;\ t \in R$ $\Rightarrow$ $\dfrac{z - z_1}{z_2 - z_1} = t =$ purely real number

19.



20. $|6z - i| \le |2 + 3iz|$ $\Rightarrow$ $|6z - i|^2 \le |2 + 3iz|^2$ $\Rightarrow$ $|z| \le \dfrac{1}{3}$

21. We have $\omega^{2n+1} = 1$ & $1 + \omega + \omega^2 + \ldots.. + \omega^{2n} = 0$

$\Rightarrow$ $1 + \omega + \omega^2 + \ldots. + \omega^n + \omega^n(\omega + \omega^2 \ldots..\ \omega^n) = 0$ $\Rightarrow$ $1 + z - \dfrac{1}{2} + \omega^n\left(z - \dfrac{1}{2}\right) = 0$

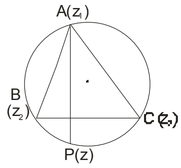
$\Rightarrow$ $(2z + 1) = -\omega^n.(2z - 1)$ $\Rightarrow$ $(2z + 1)^{2n+1} = -\omega^{n(2n+1)}(2z - 1)^{2n+1}$

$\Rightarrow$ $(2z + 1)^{2n+1} + (2z - 1)^{2n+1} = 0$ Ans. (B)

further $z = \dfrac{1}{2}.\dfrac{\omega^n - 1}{\omega^n + 1}$ $\Rightarrow$ $\bar{z} = \dfrac{1}{2}\,\dfrac{\frac{1}{\omega^n} - 1}{\frac{1}{\omega^n} + 1}$ $\Rightarrow$ $\bar{z} = -z$ $\Rightarrow$ $(\bar{z})^{2k} = z^{2k}$ & $(\bar{z})^{2k+1} = -z^{2k+1}$

22. $|z_1| = |z_2| = |z_3| = |z|$ $\Rightarrow$ $z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = z\bar{z}$

$\because$ AP $\perp^r$ BC $\therefore$ $\dfrac{z - z_1}{\bar{z} - \bar{z}_1} + \dfrac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = 0$ $\Rightarrow$ $\dfrac{z - z_1}{\dfrac{z_1\bar{z}_1}{z} - \bar{z}_1} + \dfrac{z_2 - z_3}{\dfrac{z_3\bar{z}_3}{z_2} - \bar{z}_3} = 0$

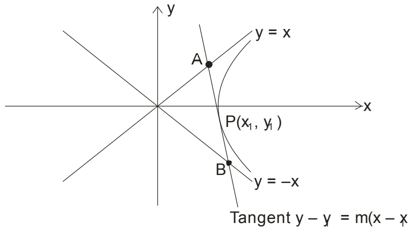


$\Rightarrow$ $-\dfrac{z}{\bar{z}_1} - \dfrac{z_2}{\bar{z}_3} = 0$ $\Rightarrow$ $z = -\dfrac{\bar{z}_1 z_2}{\bar{z}_3} = -\dfrac{z_2 z_3}{z_1}$

23. $\dfrac{dy}{dx} = \dfrac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2.1}$ $\Rightarrow$ $\dfrac{dy}{dx} = (-\cot x \pm \operatorname{cosec} x)y$

$\Rightarrow$ $\dfrac{dy}{dx} = -\cot\dfrac{x}{2}.y$ or $\dfrac{dy}{dx} = \tan\dfrac{x}{2}.y$ $\Rightarrow$ $y = c\operatorname{cosec}^2\dfrac{x}{2}$ or $y = c\sec^2\dfrac{x}{2}$

24.



$A\left(\dfrac{mx_1 - y_1}{m-1}, \dfrac{mx_1 - y_1}{m-1}\right)$ & $B\left(\dfrac{mx_1 - y_1}{m+1}, \dfrac{y_1 - mx_1}{m+1}\right)$ $\because$ P is mid point of AB

$\therefore$ $2x_1 = \dfrac{mx_1 - y_1}{m-1} + \dfrac{mx_1 - y_1}{m+1}$ $\Rightarrow$ $m = \dfrac{x_1}{y_1}$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{x}{y}$ $\Rightarrow$ $x^2 - y^2 = c$

25. Put y = tx $\Rightarrow$ $t + x\dfrac{dt}{dx} = \dfrac{t^2 - 2t - 1}{t^2 + 2t - 1}$

$\Rightarrow$ $\dfrac{-t^2 - 2t + 1}{(t+1)(t^2+1)}dt = \dfrac{dx}{x}$ $\Rightarrow$ $\left(\dfrac{1}{t+1} - \dfrac{2t}{t^2+1}\right)dt = \dfrac{dx}{x}$ $\Rightarrow$ $x^2 + y^2 = c(x + y)$

26. $(1 - x^2)\dfrac{dy}{dx} = x(1 - y)$ $\Rightarrow$ $\dfrac{dy}{y-1} = \dfrac{x}{x^2 - 1}dx$

Integrating both sides

$2\displaystyle\int \dfrac{dy}{y-1} = \int \dfrac{2x}{x^2-1}dx$ $\Rightarrow$ $2\ell n|y - 1| = \ell n|x^2 - 1| + \ell nc$ $\Rightarrow$ $(y - 1)^2 = c|x^2 - 1|$

27. $|z_1 + z_2|^2 = |z_1|^2 = |z_2|^2$ $\Rightarrow$ $|z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$

divide by $z_2\bar{z}_2$

$\dfrac{z_1}{z_2} + \overline{\left(\dfrac{z_1}{z_2}\right)} + 1 = 0$ $(\because z_1\bar{z}_1 = z_2\bar{z}_2)$ $\Rightarrow$ $\dfrac{z_1}{z_2} = \omega$ or ω^2

28. Differentiate both sides w.r.t. y, then put y = 0

$2xf'(x) - 2f(x) = 2x^2$ $\Rightarrow$ $\dfrac{dy}{dx} - \dfrac{1}{x}.y = x$ $\Rightarrow$ $y = x^2 + x$

29. $iz_2(|z_1|^2 + 1) = z_1(1 + |z_2|^2)$ $\Rightarrow$ $\dfrac{z_1}{z_2}$ = pure imaginary

further $iz_1\bar{z}_1 z_2 - z_2\bar{z}_2 z_1 = z_1 - iz_2$ $\Rightarrow$ $\bar{z}_1 z_2(iz_1 + z_2) = -i(z_2 + iz_1)$ $(\because z_1\bar{z}_2 = -\bar{z}_1 z_2)$

$\Rightarrow$ $\bar{z}_1 z_2 = -i$ or $iz_1 = -z_2$

$\Rightarrow$ $|z_1 z_2| = 1$ or $|z_1| = |z_2|$

30. (A) Standard result (B) $|1 + \alpha + \alpha^2 + \alpha^3| = |-\alpha^4| = 1$

(C) $|1 + \alpha + \alpha^2| = |-\alpha^3 - \alpha^4| = |1 + \alpha| = \left|1 + \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}\right| = 2\cos\dfrac{\pi}{5}$

31. $|z_1 + i z_2| \le |z_1| + |z_2| = 17$ Also, $|z_1 + (i+1)z_2| \ge ||z_1| - |(1+i)z_2|| = 13 - 4\sqrt{2}$

Further, $\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \frac{4}{|z_2|} = 5$ & $\left|z_2 + \frac{4}{z_2}\right| \ge \left||z_2| - \frac{4}{|z_2|}\right| = 3$

32. $\omega = \frac{1-z}{1+z} = \frac{\bar{z}-1}{\bar{z}+1} = -\overline{\left(\frac{1-z}{1+z}\right)} = -\bar{\omega}$ or $\omega + \bar{\omega} = 0$ $\Rightarrow$ ω lies on y-axis

33. to 35. $\int_0^x f(g(t))dt = \frac{1}{2}(1 - e^{-2x})$

differentiating both sides w.r.t. x

$f(g(x)) = e^{-2x}$ $\Rightarrow$ $f'(g(x)).g'(x) = -2e^{-2x}$

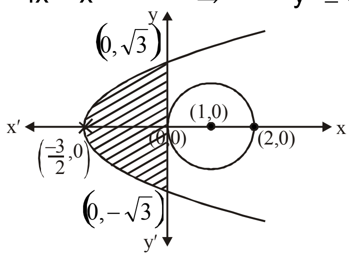
let $g(f(x)) = y$

$\therefore$ $x.y.(-2).e^{-2x} = e^{-2x}.\frac{dy}{dx}$ $\Rightarrow$ $\frac{dy}{dx} = -2yx$ $\Rightarrow$ $\frac{dy}{y} + 2x\,dx = 0$

$\Rightarrow$ $\ell n\, y + x^2 = c$ $\Rightarrow$ $y = e^{c-x^2}$ $\Rightarrow$ $g(f(x)) = e^{-x^2}$ $\therefore h(x) = \frac{e^{-x^2}}{e^{-2x}} = e^{2x-x^2}$

36. to 38. For A, $|z + 1| \le 2 + Re(z)$ $\Rightarrow$ $(x+1)^2 + y^2 \le 4 + 4x + x^2$ $\Rightarrow$ $y^2 \le 3 + 2x$

$\Rightarrow$ $y^2 \le 2\left(x + \frac{3}{2}\right)$ ...(i)



For B, $|z - 1| \ge 1$ $\Rightarrow$ $(x-1)^2 + y^2 \ge 1$ ......(2)

For C, $|z - 1|^2 \ge |z + 1|^2$ $\Rightarrow$ $x \le 0$ .....(3)

(i) (–1,0), (–1, 1) , (–1,–1), (0,0), (0,1), (0,–1) but $z \ne -1$

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 5.

(ii) Required area = $2\int_{-3/2}^{0} \sqrt{2\left(x + \frac{3}{2}\right)}\,dx = 2\sqrt{3}$

(iii) Clearly $z = \frac{-3}{2} + 0i$ is the complex number in the region $A \cap B \cap C$ having maximum amplitude. $\therefore$ $Re(z) = -3/2$

39. Let $z = e^{i\theta}$; $\theta \in [0, 2\pi)$

$\because$ $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$ $\Rightarrow$ $|2\cos 2\theta| = 1$

$\Rightarrow$ $\cos 2\theta = \pm 1/2$ $\Rightarrow$ Total 8 solutions.

40. $z_1z_2 + z_2z_3 + z_3z_1$

$= z_1z_2z_3(\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$

$\Rightarrow$ $z_1z_2 + z_2z_3 + z_3z_1 = 1$

$\therefore$ z_1, z_2, z_3 satisfy

$z^3 - z^2 + z - 1 = 0$

or $z_1 = -i$

$z_2 = 1$

$z_3 = i$ $\Rightarrow$ $|z_1 + z_2^2 + z_3^3| = |-i + 1 - i| = \sqrt{5}$